

The ζ -function answer to parity violation in three dimensional gauge theories

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Abstract

We study parity violation in $2+1$ -dimensional gauge theories coupled to massive fermions. Using the ζ -function regularization approach we evaluate the ground state fermion current in an arbitrary gauge field background, showing that it gets two different contributions which violate parity invariance and induce a Chern-Simons term in the gauge-field effective action. One is related to the well-known *classical* parity breaking produced by a fermion mass term in 3 dimensions; the other one, already present for massless fermions, is related to peculiarities of gauge invariant regularization in odd-dimensional spaces.

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1 Introduction

Gauge theories in three dimensional space-time exhibit a variety of phenomena of interest not only in Quantum Field Theory [1] but also in Condensed Matter Physics [2].

An important feature of three dimensional kinematics concerns the possibility of giving a (topological) mass to the vector field by including an unconventional term in the gauge field Lagrangian [3]-[5]: the Chern-Simons (CS) secondary characteristic.

The CS term, of topological origin, violates both P and T invariances. Since the same happens for the mass term for a (two-component) Dirac spinor in three dimensional space-time, it is natural to expect an interesting interplay between both masses in 3-dimensional gauge theories coupled to fermions. Indeed, in refs.[3]-[4],[6]-[7] it was shown that if any of the two mass terms is included in the Lagrangian, the other is then induced by radiative corrections.

Now, in ref.[6]-[7], it was also shown that even massless fermions, when coupled to gauge fields, generate a CS term. Originally this effect was thought as a consequence of the introduction of a fermion mass term within the Pauli Villars regularization procedure [7]. However, the occurrence of the CS term was confirmed in ref.[8] (hereafter referred as I) using the ζ -function approach, *where no regulating mass term is added at any stage of the calculations*. In fact, the violation of parity in odd dimensions (and, consequently, the generation of a CS term) is analogous to the non-conservation of the axial current in even dimensions: the imposition of gauge invariance produces in both cases an anomaly for a symmetry (parity, chiral symmetry) of the original action for *any* sensible regularization.

The issue of parity violation in three dimensions is relevant in different contexts. In particular, discrepancies on overall parity breaking have arisen in the analysis of the $2+1$ Thirring model [9]-[12], this leading to contradictory results concerning dynamical mass generation for fermions (see [13] for a treatment of the Thirring model using the techniques described in this paper). Since $3+1$ dimensional field theories become effectively 3-dimensional in the high temperature limit, the problem has applications also in 4 dimensional Physics. Also in Condensed Matter Physics, the appearance of Chern-Simons term through parity anomaly provides an adequate ground for testing the physics of anyons in three dimensional fermionic systems [2].

In this respect, generation of CS term through the fermion effective action has direct consequences on the thermodynamic properties of the system [14].

Since the generation of the CS term is a consequence of regularizing the three dimensional fermionic ground state current, a careful analysis of the regularization prescription is needed in order to decide whether or not parity violation occurs. To this issue we address in the present work, extending previous results that we have already presented in I.

Firstly, we give a mathematically rigorous scheme leading to the definition of the path-integral fermionic measure for a 3-dimensional gauge theory. This is achieved through a careful treatment of the fermionic determinant (a not well defined object for the unbounded Dirac operator) using the results of Seeley [15] on complex powers of elliptic operators.

Secondly, we give the recipe for computing fermionic ground state currents. Since at this stage we already dispose of a finite expression for the fermionic partition function \mathcal{Z} within the ζ -function approach, this recipe reduces to give explicit formulæ for obtaining vacuum expectation values by using the differentiability properties of the ζ -function [16].

Finally, we apply our approach to the analysis of massive fermions thus completing the work initiated in I on massless fermions. We think that the present analysis clarifies the origin of the parity anomaly and, in particular, the appearance of two contributions to parity violation, one originated in peculiarities of calculations in odd-dimensional spaces, the other arising from the parity violation produced, already at the classical level, by the fermion mass in 3 dimensions.

In respect with massless fermions, the original calculation of the ground state fermion current $j_\mu[A]$ in a constant magnetic background $*F_\mu$, $*F_\mu = i\epsilon_{\mu\nu\alpha}\partial_\nu A_\alpha$ was presented in ref.[7] (we are working here in Euclidean space). As an example, the perturbative calculation using Pauli-Villars regularization yields for the Abelian case, in the one loop approximation [7],

$$j_\mu[A] = \frac{m}{|m|} \frac{e}{4\pi} *F_\mu \quad (1)$$

where e is the gauge coupling constant and m is a fermionic mass to be put to zero at the end of the calculations. This current is identically conserved, but it violates parity conservation since $*F_\mu$ is a pseudovector.

From this current the effective action $\Gamma[A]$ can be computed, using the relation

$$\frac{\delta\Gamma[A]}{\delta A_\mu} = ej_\mu[A]. \quad (2)$$

From eqs.(1)-(2) we see that the effective action, appart from parity conserving contributions, contains a CS term,

$$\Gamma[A]_{Pauli-Villars} = \frac{m}{|m|} \frac{e^2}{2} S_{CS}[A] + S_{PC}[A] \quad (3)$$

where $S_{CS}[A]$ is the (Abelian) Chern-Simons action,

$$S_{CS}[A] = \frac{1}{4\pi} \int d^3x A_\mu {}^*F_\mu \quad (4)$$

and $S_{PC}[A]$ are parity conserving terms. We have explicitly indicated in Γ that this result was obtained in [7] within the Pauli-Villars regularization scheme.

Of course, this result can be trivially generalized to the case of N flavors. Instead of (3), one has when there are N fermion species:

$$\Gamma[A]_{Pauli-Villars} = \sum_{i=1}^N \frac{m_i}{|m_i|} \frac{e^2}{2} S_{CS}[A] + S_{PC}[A, N]. \quad (5)$$

As in the $N = 1$ case, one has to put the fermion masses $m_i = 0$ to recover the massless case. Within this approach, one could evidently choose the signs of m_i 's in such a way that, for even N , the overall CS contribution to Γ cancels out so that overall parity would not be violated. Now, using the ζ -function approach, we have proven in I, without necessity of introducing a mass-parameter, that instead of (3) one directly obtains:

$$\Gamma[A]_\zeta = \pm \frac{e^2}{2} S_{CS}[A] + S_{PC}[A] \quad (6)$$

As it will become clear in the next section, within the ζ -function approach, the sign ambiguity in (6) can be traced back to the choice of an integration path Ω in the complex plane, necessary for defining the complex powers of the Dirac operator. In odd-dimensional space-times, the choice of Ω in the upper (lower) half plane yields to a positive (negative) sign in (6). Then, once

a definite integration path is chosen, consistency in the definition of complex powers implies that, when N species are present, all fermions contribute with the same sign so that, instead of eq.(5) one gets

$$\Gamma[A]_{\zeta} = \pm N \frac{e^2}{2} S_{CS}[A] + S_{PC}[A, N] \quad (7)$$

and overall parity is always violated.

To end with the resumé of the massless case analysis, it should be stressed that the ζ -function approach allows the calculation of the parity violating contribution to $j_{\mu}[A]$ for *arbitrary* A_{μ} (and not necessarily for one leading to a constant magnetic field) in an *exact* form. As explained above, it was shown in Ref.[7] that the parity violation contribution was induced from one fermion loops and arguments were given to discard higher loop corrections. Now, we have proven in I that the Chern-Simons contribution present in (6) is *all* the parity violating contribution one has to expect for three dimensional massless fermions since, as in the case of the chiral anomaly, the ζ -function approach gives the exact anomalous contribution.

Let us come now to the massive fermion case. The problem was already discussed in [3], to the one-loop order and in a constant magnetic background, using Pauli-Villars regularization and clearly explained in [1]. The conclusion was that when massive fermions couple to the gauge field, a CS term is induced by fermion radiative corrections. The fact that a fermion mass term in three dimensions violates P and T was considered at the origin of this result. The explicit form for the effective action in the massive case was calculated in [7]. The result coincides with (3) except that in this case m is the physical fermion mass. Again, the conclusion was dependent on how to regularize divergent objects [3],[18]-[19]. It is then worthwhile to analyse the problem of massive fermions using the ζ -function approach developed in I. We here undertake such program explaining in section 2 the mathematical tools needed for the computation of ground state fermion currents and leaving for section 3 the explicit calculation of j_{μ} . A summary of results and discussions is given in Section 4.

2 How to compute currents

We first consider the Abelian case giving the basic formulæ to be used in the next section.

The partition function for a massive Dirac fermion doublet in $d=3$ Euclidean space-time dimensions is:

$$\mathcal{Z}[A_\mu] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp[-\int \bar{\psi} D[A] \psi d^3x] \quad (8)$$

where A_μ is a background vector field and the Dirac operator $D[A]$ is given by¹

$$D[A] = \gamma_\mu (i\partial_\mu + eA_\mu) + im. \quad (9)$$

In (8), $\mathcal{D}\bar{\psi} \mathcal{D}\psi$ is some fermionic measure to be defined below. The results we shall describe are valid for any elliptic operator L , not necessarily Hermitian, defined on a compact manifold without boundary [16],[17]. We shall assume however that they are also valid for R^3 .

The ground state current $j_\mu[A]$ in the presence of the background field,

$$j_\mu[A] = \langle \bar{\psi}(x) \gamma_\mu \psi(x) \rangle_A, \quad (10)$$

can be calculated from $\mathcal{Z}[A]$ as

$$j_\mu[A] = -\frac{1}{e} \frac{\delta}{\delta A_\mu(x)} \log \mathcal{Z}[A]. \quad (11)$$

Of course, since the Dirac operator is unbounded, $\mathcal{Z}[A]$ needs a regularization,

$$\mathcal{Z}_{reg}[A] \equiv \det D[A] \Big|_{reg}. \quad (12)$$

We shall adopt in this work the ζ -function regularization method which automatically ensures *local gauge invariance*. We thus define as usual

$$\zeta(D[A], s) = \text{Tr}(D^{-s}[A]), \quad (13)$$

since $D[A]$ is invertible. From (13) we can define

$$\mathcal{Z}_{reg}[A] = \exp\left(-\frac{d}{ds} \zeta(D[A], s)\right) \Big|_{s=0} \quad (14)$$

¹Our conventions for γ matrices are $\gamma_\mu = \sigma_\mu$ with σ_μ ($\mu = 1, 2, 3$) the Pauli matrices.

where the right-hand side should be computed for complex s (with large enough real part) and then analytically extended to $s = 0$. This definition amounts in practice to the definition of the fermionic path-integral measure. It has been proven in [16] that $\mathcal{Z}[A]$ written as in eq.(14) is differentiable; then, one can easily obtain from eqs.(11),(14) a regularized expression for $j_\mu[A]$. Indeed, one can write

$$\int \frac{\delta}{\delta A_\mu(x)} \log \mathcal{Z}_{reg}[A] a_\mu(x) d^3x = \frac{d}{dt} \log \mathcal{Z}_{reg}[A_\mu + t a_\mu] \Big|_{t=0}. \quad (15)$$

(Here a_μ indicates a direction in A_μ functional space along which the derivative is taken). Now, from eq.(14) one has

$$\int \frac{\delta}{\delta A_\mu(x)} \log \mathcal{Z}_{reg}[A] a_\mu(x) d^3x = \frac{d}{dt} \left(-\frac{d}{ds} \zeta(D[A_\mu + t a_\mu], s) \Big|_{s=0} \right) \Big|_{t=0} \quad (16)$$

or, using eq.(13) and interchanging the order of the derivatives,

$$\int \frac{\delta}{\delta A_\mu(x)} \log \mathcal{Z}_{reg}[A] a_\mu(x) d^3x = \frac{d}{ds} \left[s \text{Tr}(D^{-s-1}[A] e \gamma_\mu a_\mu) \right] \Big|_{s=0}. \quad (17)$$

In eqs.(13),(17), “Tr” stands for the operator trace, which includes the matrix trace “tr” and an integration over the space-time of the diagonal of the operator kernel. If we denote by $K_s(x, y; D[A])$ the kernel of the operator $D^s[A]$,

$$D^s[A] f(x) = \int d^3y K_s(x, y; D[A]) f(y) \quad (18)$$

we can read from eq.(17)

$$\frac{\delta}{\delta A_\mu(x)} \log \mathcal{Z}_{reg}[A] = e \frac{d}{ds} \left\{ s \text{Tr} [\gamma_\mu K_{-s-1}(x, x; D[A])] \right\} \Big|_{s=0}. \quad (19)$$

Then, from eqs.(11)-(19), one has

$$j_\mu^{reg}[A] = -\frac{d}{ds} \left\{ s \text{Tr} [\gamma_\mu K_{-s-1}(x, x; D[A])] \right\} \Big|_{s=0}. \quad (20)$$

This is the key formula we shall employ to compute fermionic currents in the ζ -function regularization scheme.

Note that $K_{-s-1}(x, y; D[A])|_{s=0}$ is nothing but the Green function $G(x, y)$ of the operator $D[A]$ when $x \neq y$. Correspondingly, it is singular on the diagonal $x = y$. Now, for any elliptic operator L of positive order r in a space of dimension n , the kernel of L^s is a continuous function of x, y for $\text{Re}(s) < -n/m$ that admits an analytic extension to the whole complex plane s if $x \neq y$. On the diagonal $x = y$ it is a meromorphic function of s that has at most simple poles at $s = (-n + j)/r$, $j = 0, 1, \dots$ [15]. For the Dirac operator $D[A]$ in three-dimensional Euclidean space-time, $K_{-s-1}(x, x; D[A])$ has a simple pole at $s = 0$. It is then clear that eq.(20) gives j_μ^{reg} as the finite part of the Laurent series for $-tr[\gamma_\mu K_{-s-1}(x, x; D[A])]$ at $s = 0$, thus providing an operative (finite) formula for $-tr[\gamma_\mu G(x, x)]$

$$-\frac{d}{ds} \left\{ s \text{tr} [\gamma_\mu K_{-s-1}(x, x; D[A])] \right\} \Big|_{s=0} = -tr[\gamma_\mu G^{reg}(x, x)]. \quad (21)$$

The right hand side in (21) gives the ζ -function analogous of the usual expression that can be found in the literature [20] for the ground state current, with the regularization procedure specified in the present approach by the left hand side.

We see at this point that for computing $j_\mu^{reg}[A]$ we need an explicit formula for the kernel $K_{-s-1}(x, y; D[A])$ so as to evaluate the r.h.s. of eq.(20). For the sake of clarity and consistency, let us present some of the results of Seeley [15] on complex powers of elliptic operators in the restricted form needed for studying the Dirac operator.

The complex powers of the elliptic differential operator $D[A]$ (of order 1 in a space of dimension $n=3$) are best described in terms of the symbol of the operator and the symbol of its resolvent $(D[A] - \lambda)^{-1}$. The symbol of $D[A]$ is a polynomial $\sigma(D[A])$ in a vector ξ_μ that can be thought as the Fourier variable associated to x_μ . It is obtained from $D[A]$ by replacing $-i\partial_\mu \rightarrow \xi_\mu$ and takes the form

$$\sigma(D[A])(x, \xi) = -\gamma_\mu \xi_\mu + e\gamma_\mu A_\mu(x) + im \equiv a_1(x, \xi) + a_0(x, \xi) \quad (22)$$

where $a_j(x, \xi)$ ($j=0,1$) are homogeneous functions of degree j in ξ_μ .

The resolvent $(D[A] - \lambda)^{-1}$ of the differential operator $D[A]$ is a pseudo-differential operator [21]. Its symbol is a generalization of the definition above and can be properly approximated by

$$\sigma((D[A] - \lambda)^{-1})(x, \xi) = \sum_{j=0}^{\infty} b_{-1-j}(x, \xi; \lambda). \quad (23)$$

Here $b_{-1-j}(x, \xi; \lambda)$ are homogeneous functions of degree $-1-j$ in ξ_μ and λ . These coefficients can be recursively evaluated for each order in ξ_μ from the relation²

$$\sigma((D[A] - \lambda)^{-1}) \circ \sigma((D[A] - \lambda)) = I \quad (24)$$

provided one takes λ order ξ , so that it combines with the top term a_1 of the symbol of $D[A]$. The recursive formula reads

$$b_{-1}(a_1 - \lambda) = I, \quad (25)$$

$$b_{-1-l}(a_1 - \lambda) + \sum_{j=0}^{l-1} \sum_{\alpha=0}^{l-j} \frac{1}{\alpha!} \left(\frac{\partial}{\partial \xi} \right)^{\vec{\alpha}} b_{-1-j} \left(-i \frac{\partial}{\partial x} \right)^{\vec{\alpha}} a_{1-l+\alpha+j} = 0, \quad l=1,2,\text{Dots}$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is a vector of non negative integers, $\alpha = \sum_{i=1}^3 \alpha_i$ and $(\frac{\partial}{\partial \xi})^{\vec{\alpha}} = \prod_{\mu=1}^3 (\frac{\partial}{\partial \xi_\mu})^{\alpha_\mu}$, $(\frac{\partial}{\partial x})^{\vec{\alpha}} = \prod_{\mu=1}^3 (\frac{\partial}{\partial x_\mu})^{\alpha_\mu}$.

From eqs.(22),(25) the coefficients of the symbol of the resolvent can be evaluated for the Dirac operator. The first three of them are

$$b_{-1}(x, \xi; \lambda) = \frac{1}{\lambda^2 - \xi^2} (\not{x} - \lambda), \quad (26)$$

$$b_{-2}(x, \xi; \lambda) = -\frac{1}{(\lambda^2 - \xi^2)^2} (\not{x} - \lambda)(e \not{A} + im)(\not{x} - \lambda), \quad (27)$$

$$b_{-3}(x, \xi; \lambda) = \frac{1}{(\lambda^2 - \xi^2)^3} (\not{x} - \lambda)(e \not{A} + im)(\not{x} - \lambda)(e \not{A} + im)(\not{x} - \lambda) + ie \frac{\partial_\mu A_\nu}{(\lambda^2 - \xi^2)^3} (2\xi_\mu (\not{x} - \lambda) \gamma_\nu (\not{x} - \lambda) + (\lambda^2 - \xi^2) \gamma_\mu \gamma_\nu (\not{x} - \lambda)). \quad (28)$$

Following Seeley [15], the complex powers $D^s[A]$ can be defined as a generalization of Cauchy integral representation of z^s ,

$$z^s = \frac{i}{2\pi} \oint_C \frac{\lambda^s}{z - \lambda} d\lambda. \quad (29)$$

Here, the contour C must avoid a log-like cut in the complex plane λ and encircle the pole of the integrand. For the operator $D[A]$, one must take a

²The product of symbols is defined in such a way that that it corresponds to the symbol of the composition of operators [21].

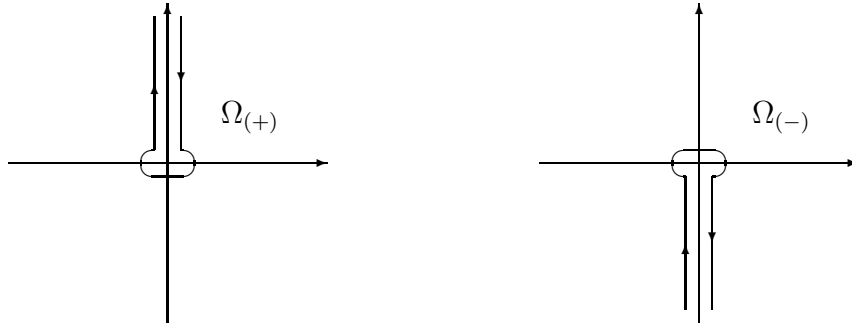


Figure 1: Contours Ω for defining complex powers of $D[A]$.

contour Ω that encircles the whole spectrum of $D[A]$ and avoids a ray going from the origin to infinity in a direction such that no eigenvalue of a_1 lies on it (such a ray is called a ray of minimal growth or Agmon ray). For $\text{Re}(s) < 0$ one can take Ω as a curve beginning at ∞ , passing along the ray of minimal growth to a small circle about the origin, then clockwise around the circle, and back to ∞ along the ray. The ray along which curve Ω is defined cannot coincide with the real axis, since being $D[A]$ Hermitian its eigenvalues are real. Two possible Ω curves are illustrated in Fig. 1.

Then, for $\text{Re}(s) < 0$, one defines

$$D^s[A] = \frac{i}{2\pi} \int_{\Omega} \lambda^s (D[A] - \lambda)^{-1} d\lambda. \quad (30)$$

This definition is analytically extended to the whole complex plane through multiplication by integer powers of $D[A]$,

$$D^s[A] = D^k[A] D^{s-k}[A] \quad k \text{ integer}, \quad -1 \leq \text{Re}(s) - k < 0. \quad (31)$$

The corresponding symbol is constructed using eq.(23) and reads

$$\sigma(D^s[A])(x, \xi) = \sum_{j=0}^{\infty} \frac{i}{2\pi} \int_{\Omega} \lambda^s b_{-1-j}(x, \xi; \lambda) d\lambda \equiv \sum_{j=0}^{\infty} C_j(x, \xi; s), \quad (32)$$

where

$$C_j(x, \xi; s) = \frac{i}{2\pi} \int_{\Omega} \lambda^s b_{-1-j}(x, \xi; \lambda) d\lambda \quad (33)$$

are homogeneous functions of (complex) degree $s - j$. It is worthwhile noting that, while eq.(30) is valid for $Re(s) < 0$, the relation between symbols eq.(32) is valid for any value of s .

Using the tools described above we are now in conditions to write the kernel $K_{-s-1}(x, \xi; D[A])$ for the operator $D^{-s-1}[A]$, necessary for evaluating the ground state current (eq.(20)). Indeed, one of the main points in Seeley's work [15] is that one can approximate kernel K_s through Fourier transforms of the C_j coefficients,

$$K_s(x, y; D[A]) = \sum_{j=0}^N \int \frac{d\xi}{(2\pi)^n} C_j(x, \xi; s) e^{i(x-y) \cdot \xi} + R(x, y; s). \quad (34)$$

Here $R(x, y; s)$ includes terms which will not contribute to the derivative in eq.(20) at $s = 0$ if N is taken sufficiently large. Though the integral in each term converges only for $Re(s) < (-n + j)$ (being n the dimension of spacetime, here $n=3$), this expression can be analytically extended to arbitrary s for $x \neq y$. On the diagonal $x = y$, $K_s(x, x; D[A])$ is extended to a meromorphic function with at most simple poles at $s = (-n + k)$, $k = 0, 1, \dots$, one arising from each term included in the sum in eq.(34).

In order to compute j_{μ}^{reg} in eq.(20) one uses eq.(34) to evaluate the finite part of the simple pole of $K_s(x, x; D[A])$ at $s = -1$. To this end, we shall use the following proposition (proven in I and adapted here to the case of the Dirac operator):

Proposition: For the elliptic invertible first order operator $D[A]$ on a 3-dimensional compact manifold M without boundary, the following identity holds:

$$\left. \frac{d}{ds} (s K_{s-1}(x, x; D[A])) \right|_{s=0} =$$

$$\begin{aligned}
& \lim_{y \rightarrow x} \left\{ G(x, y) - \int \frac{d^3 \xi}{(2\pi)^3} C_1(x, \xi/|\xi|; -1) |\xi|^{-2} e^{i\xi \cdot (x-y)} \right. \\
& \quad - \int \frac{d^3 \xi}{(2\pi)^3} C_0(x, \xi/|\xi|; -1) |\xi|^{-1} e^{i\xi \cdot (x-y)} - \int_{|\xi| \geq 1} \frac{d^3 \xi}{(2\pi)^3} C_2(x, \xi; -1) e^{i\xi \cdot (x-y)} \Big\} \\
& \quad - \int_{|\xi|=1} \frac{d}{ds} C_2(x, \xi; s) \Big|_{s=-1} \frac{d^2 \xi}{(2\pi)^3} \tag{35}
\end{aligned}$$

where $G(x, y)$ is the Green function of $D[A]$. (The interested reader can find the proof in I.)

The last term between brackets in eq.(35) can be conveniently rewritten as [22] (see I)

$$\begin{aligned}
& \int_{|\xi| \geq 1} \frac{d^3 \xi}{(2\pi)^3} C_2(x, \xi; -1) e^{i\xi \cdot (x-y)} = \\
& \quad h_0(x, x-y) + M(x)(\log |x-y| + C(x-y)) \tag{36}
\end{aligned}$$

where $h_0(x, z)$ is a homogeneous function of degree zero defined as

$$h_0(x, z) = \int \frac{d^3 \xi}{(2\pi)^3} \text{P.V.} [C_2(x, \xi/|\xi|; -1) - M(x)] |\xi|^{-3} e^{i\xi \cdot z}, \tag{37}$$

$M(x)$ is defined as

$$M(x) = \frac{1}{\omega} \int_{|\xi|=1} C_2(x, \xi; -1) d^2 \xi \tag{38}$$

and finally

$$\omega(\log |z| + C(z)) = - \int_{|\xi| \geq 1} \frac{d^3 \xi}{(2\pi)^3} |\xi|^{-3} e^{-i\xi \cdot z} \tag{39}$$

where $C(z)$ is a regular function in the neighborhood of $z = 0$. All Fourier transforms are taken in the sense of distributions and P.V. means principal value.

The interpretation of eq.(35) is as follows: The last three terms between brackets subtract the singularities of $G(x, y)$ on the diagonal $x = y$. Concerning the last term in eq.(35), it is a *local* regular term generated through

the regularization procedure that adds to the regular part of $G(x, x)$. In view of this interpretation we rewrite the result in eq.(35) in the form

$$\left. \frac{d}{ds}(sK_{s-1}(x, x; D[A])) \right|_{s=0} = G^{reg}(x, x) = G^{(subst)}(x) + G^{(local)}(x) \quad (40)$$

where

$$G^{(subst)}(x) = \lim_{y \rightarrow x} \left\{ G(x, y) - \int \frac{d^3\xi}{(2\pi)^3} C_1(x, \xi/|\xi|; -1) |\xi|^{-2} e^{i\xi \cdot (x-y)} \right. \\ \left. - \int \frac{d^3\xi}{(2\pi)^3} C_0(x, \xi/|\xi|; -1) |\xi|^{-1} e^{i\xi \cdot (x-y)} - \int_{|\xi| \geq 1} \frac{d^3\xi}{(2\pi)^3} C_2(x, \xi; -1) e^{i\xi \cdot (x-y)} \right\} \quad (41)$$

and

$$G^{(local)}(x) = - \int_{|\xi|=1} \frac{d}{ds} C_2(x, \xi; s) \Big|_{s=-1} \frac{d^2\xi}{(2\pi)^3} \quad (42)$$

In spite of its apparent complexity, the result in eq.(35) can be used with great simplicity for evaluating ground state currents *for arbitrary gauge background fields*. This will be seen in the next section, where we shall employ eqs.(20),(35) to compute the parity violating terms of $j_\mu^{reg}[A]$.

3 The current for massive fermions

Formula (35) is at the root of the ζ function regularization prescription for the calculation of ground state fermion currents. Using this approach, parity violating contribution to the ground state current was evaluated in I for the case of massless fermions. In this section the analysis is extended to the case of massive fermions with partition function given by eq.(8).

Let us start by understanding why parity violating terms should be expected in a 2+1 fermionic theory. First, note that a 3-dimensional fermion mass term constructed from two-component spinors violates parity and time-inversion. Indeed, under parity transformation the vector and fermion fields behave as

$$\begin{aligned} \mathcal{P} A_0(x) \mathcal{P}^{-1} &= A_0(x') \\ \mathcal{P} A_1(x) \mathcal{P}^{-1} &= -A_1(x') \end{aligned}$$

$$\begin{aligned}
\mathcal{P}A_2(x)\mathcal{P}^{-1} &= A_2(x') \\
\mathcal{P}\psi(x)\mathcal{P}^{-1} &= \sigma^1\psi(x')
\end{aligned}
\tag{43}$$

with

$$\begin{aligned}
x &= (x_0, x_1, x_2) \\
x' &= (x_0, -x_1, x_2),
\end{aligned}
\tag{44}$$

so that the fermion mass term changes sign under parity,

$$\mathcal{P}m\bar{\psi}(x)\psi(x)\mathcal{P}^{-1} = -m\bar{\psi}(x')\psi(x'). \tag{45}$$

The same happens with time inversion. Analogous results can be derived for any odd dimensional fermionic fields [23].

Using eqs.(43)-(44) one can easily show that the Dirac operator Green function changes under parity as follows

$$\mathcal{P}G_{(m)}(x, y)\mathcal{P}^{-1} = -\sigma^1 G_{(-m)}(x', y')\sigma^1 \tag{46}$$

This formula (46) is only meaningful for $x \neq y$, where the Green function is well-defined (we indicate with the subscript (m) the fact we are considering the massive Dirac operator case). Let us now define the object $\mathcal{J}_{(m)}^\mu(x, y)$,

$$\mathcal{J}_{(m)}^\mu(x, y) = \text{tr} \gamma^\mu G_{(m)}(x, y) \tag{47}$$

Again, this object is well-defined whenever $x \neq y$. At $x = y$, precisely where it gives the ground state current (see eq.(21)), it has to be regulated since, apart from regular terms, it has divergent contributions.

Eq.(46) shows that only for $m = 0$ $\mathcal{J}_{(m)}^\mu(x, y)$ behaves as a vector

$$\begin{aligned}
\mathcal{P}\mathcal{J}_{(0)}^0(x, y)\mathcal{P}^{-1} &= \mathcal{J}_{(0)}^0(x', y') \\
\mathcal{P}\mathcal{J}_{(0)}^1(x, y)\mathcal{P}^{-1} &= -\mathcal{J}_{(0)}^1(x', y') \\
\mathcal{P}\mathcal{J}_{(0)}^2(x, y)\mathcal{P}^{-1} &= \mathcal{J}_{(0)}^2(x', y')
\end{aligned}
\tag{48}$$

(Again, identities (48) make sense only for $x \neq y$). Then, as argued in I, parity violating contributions to the ground state current for *massless fermions* cannot arise from regular terms in $\mathcal{J}_{(0)}^\mu(x, x)$ since these terms should satisfy transformation law (48). Only additional regular terms generated by the

ζ -function prescription in the process of giving meaning to $\mathcal{J}_{(m)}^\mu(x, x)$ can break in this case parity invariance: they are not subject to (48) but just to respect gauge invariance. This is the reason why we were able to compute in I the *complete* parity violating contribution to $j_\mu(x)$ for arbitrary A_μ : it was not necessary to have a complete knowledge of $G_{(0)}(x, y)$ for arbitrary A_μ but just the behavior of the additional regular terms introduced by the ζ -function prescription. One should interpret this parity anomalous contribution present even for massless fermions as a consequence of regularization in odd dimensional space-time.

It is clear from eq.(47) that the massive case is more involved: appart from the additional terms generated by the ζ -function regularization, regular terms which are already present in $G_{(m)}(x, x)$ (and a fortiori in $\mathcal{J}_{(m)}^\mu(x, x)$) may in principle contribute to parity violation since they are not constrained to satisfy vector-like parity transformations. This in turn implies that, in the massive case, one needs a more detailed knowledge of $G_{(m)}(x, y)$. For that reason we shall have to limit the validity of ours results to the domain of a perturbative expansion. With this in mind, we rewrite eq.(20) as

$$j_\mu^{reg}[A] = -tr \left[\gamma_\mu \frac{d}{ds} (sK_{s-1}(x, x; D[A])) \right]_{s=0} \quad (49)$$

and use the proposition presented in the previous section in the form given by (40)-(42).

We start analysing the regular term in eq.(40) generated by the ζ -function method,

$$G^{(local)}(x) = - \int_{|\xi|=1} \frac{d}{ds} C_2(x, \xi; s) \Big|_{s=-1} \frac{d^2 \xi}{(2\pi)^3}. \quad (50)$$

Using the definition for C_2 we have, after differentiating,

$$G^{(local)}(x) = - \frac{i}{2\pi} \int_{|\xi|=1} \left[\int_\Omega \frac{\ln \lambda}{\lambda} b_{-3}(x, \xi; \lambda) d\lambda \right] \frac{d^2 \xi}{(2\pi)^3}. \quad (51)$$

As the ξ -integral is extended to S_2 , one can see that only terms even in ξ will give non vanishing contributions to $G^{(local)}(x)$. After some algebra one gets from eq.(28)

$$G^{(local)}(x) = - \frac{i}{2\pi} \int_{|\xi|=1} \frac{d^2 \xi}{(2\pi)^3} \int_\Omega d\lambda \frac{\ln \lambda}{\lambda(\lambda^2 - 1)^3}$$

$$\begin{aligned}
& \left[\lambda(e^2 A^2 - 4e^2 A_\mu A_\nu \xi_\mu \xi_\nu + 3m^2 + 2iem \not{A} - 8iem A_\mu \xi_\mu \not{\xi} \right. \\
& + ie \partial_\mu A_\nu (-4\xi_\mu \xi_\nu + \delta_{\mu\nu} + i\epsilon_{\mu\nu\alpha} \gamma_\alpha)) \\
& \left. + \lambda^3 (-e^2 A^2 - 2iem \not{A} + m^2 - ie \partial_\mu A_\nu (\delta_{\mu\nu} + i\epsilon_{\mu\nu\alpha} \gamma_\alpha)) \right] \quad (52)
\end{aligned}$$

We have at this point to choose a curve Ω in order to perform the λ integral, this choice determining the branch for the log function in eq.(52). Let us first consider the curve $\Omega_{(+)}$ depicted in Fig. 1 and call

$$I_{(+)}[p, q] = \int_{\Omega_{(+)}} \frac{\lambda^q \ln \lambda}{(\lambda^2 - 1)^p} d\lambda. \quad (53)$$

It is easy to see that the integration along the small circle around the origin vanishes for $q \geq 0$ as its radius goes to zero while the integrals along the ray sum up to give

$$I_{(+)}[p, q] = i^q (-1)^p \pi B\left(\frac{q+1}{2}, p - \frac{q+1}{2}\right) \quad (54)$$

where $B(m, n)$ is the β function (Euler's integral of the first kind) [24] defined as

$$B(m, n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt. \quad (55)$$

The β functions can be evaluated in terms of Γ functions to give

$$I_{(+)}[p, q] = i^q (-1)^p \pi \frac{\Gamma(\frac{q+1}{2}) \Gamma(p - \frac{q+1}{2})}{\Gamma(p)}. \quad (56)$$

Using this result and performing the remaining ξ integral one gets

$$G_{(+)}^{(local)}(x) = -\frac{ie}{8\pi} \epsilon_{\mu\nu\alpha} \partial_\mu A_\nu \gamma_\alpha + \frac{i}{4\pi} m^2 \quad (57)$$

where the subscript $(+)$ indicates that we have used the $\Omega_{(+)}$ contour. The corresponding contribution to the fermionic current (49) is

$$j_{\mu(+)}^{(local)} = -tr[\gamma_\mu G_{(+)}^{(local)}(x)] = \frac{ie}{4\pi} \epsilon_{\alpha\beta\mu} \partial_\alpha A_\beta. \quad (58)$$

If instead one chooses the curve Ω_- (see Fig. 1) one gets the same expression but with the opposite sign so we finally have for the regular term generated by the ζ -function method

$$j_{\mu\pm}^{(local)} = \pm \frac{ie}{4\pi} \epsilon_{\alpha\beta\mu} \partial_\alpha A_\beta. \quad (59)$$

Let us first discuss in more detail the origin of the sign ambiguity in (59). As discussed in I, $I_\Omega[p, q]$ in eq.(53) satisfies

$$I_{\Omega_{(-)}}[p, q] = (-1)^{q+1} I_{\Omega_{(+)}}[p, q] \quad (60)$$

where $\Omega_{(+)}$ is any curve that avoids a ray of minimal growth of $D[A]$ on the upper half-plane and $\Omega_{(-)}$ is any other curve that avoids such a ray on the lower half-plane. Now, for odd dimensional spaces, only even values for q arise. In the present $d = 3$ case $q = 0, 2$. On the other hand, for even dimensional spaces $q = 2k + 1$. This is the reason why there are sign ambiguities in computing anomalies in odd dimensional spaces (according to the choice of curves Ω), which are not present for even dimensions.

Another important lesson learnt in this calculation is that this result is identical to that corresponding to massless fermions. Indeed, *no new terms* arise from $G_{(\pm)}^{(local)}$ due to the presence of the mass term since, as it happens when computing the chiral anomaly [17], the trace in (49) makes all the terms proportional to m and m^2 vanish.

We thus see that the contribution to parity violation in $j_\mu^{reg}[A]$ arising from the regular term generated by the ζ -function method is independent of whether the fermions are massless or massive (even if the mass is a regulating mass that should be taken to zero at the end of the calculation). Its origin can be traced back to gauge invariant regularization in odd dimensional spaces and it is present both in the massless and massive case.

We have now to consider regular terms coming from the brackets in eq.(35), that is, the regular part of $G_{(m)}(x, x)$ remaining once the adequate subtractions are performed. As stated above, there was no contribution to parity violation from these terms in the massless case due to the transformation law (46) obeyed by $G_{(0)}(x, x)$: parity was respected at the classical level so that violations cannot arise in the process of subtracting singular terms to $G_{(0)}(x, x)$. On the contrary, there may be parity violating contributions in the case of the massive Green function since already at the classical level parity is violated by the massive Dirac operator. Should these contributions exist, they will be the product of parity non-invariance introduced in the Lagrangian through the fermionic mass term. To see whether this happens, we are faced to the necessity of making some sort of approximation to compute

$G_{(m)}(x, y)$ at $x = y$ and, from it, the regular part of $G_{(m)}(x, x)$ defined by $G^{(subst)}(x)$ in eq.(41).

Of course, for $A_\mu = 0$ the Green function can be computed closely,

$$G_{(m)}(x, y; A_\mu = 0) = \frac{1}{4\pi} (i\not{\partial} - im) \frac{e^{-|m||x-y|}}{|x-y|}. \quad (61)$$

It is then natural to seek for a perturbative expansion in powers of the coupling constant e ,

$$G_{(m)}(x, y; A_\mu) = G_{(m)}(x, y; A_\mu = 0) - e \int d^3w G_{(m)}(x, w; A_\mu = 0) \not{A}(w) G_{(m)}(w, y; A_\mu = 0) + O(e^2) \quad (62)$$

We will keep only the first order in the perturbative expansion, but a systematic perturbative calculation can be straightforwardly implemented. In order to analyse an arbitrary gauge field configuration, we will make a Taylor expansion of $A_\mu(w)$ around x . This leads to a derivative expansion which for dimensional analysis takes the form of a ∂/m expansion [25]. Up to first derivatives the Green function can be expressed as a Laurent expansion in $z_\mu = y_\mu - x_\mu$ as

$$\begin{aligned} G_{(m)}(x, y; A_\mu) = & \frac{1}{4\pi} \left\{ i \frac{\not{z}}{|z|^3} - im \frac{1}{|z|} - \frac{im^2}{2} \frac{\not{z}}{|z|} + im|m| \right. \\ & + e A_\alpha(x) \frac{z_\alpha \not{z}}{|z|^3} - em A_\alpha(x) \frac{z_\alpha}{|z|} \\ & \left. - \frac{ie}{2} \frac{m}{|m|} \epsilon_{\mu\nu\alpha} \partial_\mu A_\nu(x) \gamma_\alpha + \frac{ie}{2} \epsilon_{\mu\nu\alpha} \partial_\mu A_\nu(x) \frac{z_\alpha}{|z|} + \frac{e}{2} \partial_\mu A_\nu(x) \frac{z_\mu z_\nu \not{z}}{|z|^3} \right\} + O(z). \end{aligned} \quad (63)$$

As expected, non-regular terms do appear on the diagonal of the Green function ($z_\mu \rightarrow 0$). According to Proposition (35) these terms should be cancelled by the subtractions. Indeed, the explicit values of these subtractions are

$$\int \frac{d^3\xi}{(2\pi)^3} C_1(x, \xi/|\xi|; -1) |\xi|^{-2} e^{i\xi \cdot (x-y)} = \frac{1}{4\pi} i \frac{\not{z}}{|z|^3}, \quad (64)$$

$$\int \frac{d^3\xi}{(2\pi)^3} C_0(x, \xi/|\xi|; -1) |\xi|^{-1} e^{i\xi \cdot (x-y)} = \frac{1}{4\pi} \left\{ -im \frac{1}{|z|} + e A_\alpha(x) \frac{z_\alpha \not{z}}{|z|^3} \right\}, \quad (65)$$

and

$$\begin{aligned} \int_{|\xi| \geq 1} \frac{d^3 \xi}{(2\pi)^3} C_2(x, \xi; -1) e^{i\xi \cdot (x-y)} &= \frac{1}{4\pi} \left\{ -\frac{im^2}{2} \frac{\not{z}}{|z|} - em A_\alpha(x) \frac{z_\alpha}{|z|} \right. \\ &\quad \left. + \frac{ie}{2} \frac{m}{|m|} \epsilon_{\mu\nu\alpha} \partial_\mu A_\nu(x) \frac{z_\alpha}{|z|} + \frac{e}{2} \partial_\mu A_\nu(x) \frac{z_\mu z_\nu \not{z}}{|z|^3} \right\} + O(e^2). \end{aligned} \quad (66)$$

so that using (35) non-regular terms are exactly cancelled. In the last expression we used eqs.(36)-(38) where $M(x) = 0$; by $O(e^2)$ we just mean some non-regular terms proportional to e^2 . One can check that each subtraction corresponds to non-regular terms homogeneous of degree -2, -1 and 0 respectively in z_μ .

The regular part of $G_{(m)}(x, x)$ remaining after subtractions and the limit $y \rightarrow x$ are performed is

$$G^{(subst)}(x) = \frac{1}{4\pi} \{ im|m| - \frac{ie}{2} \frac{m}{|m|} \epsilon_{\mu\nu\alpha} \partial_\mu A_\nu(x) \gamma_\alpha \}. \quad (67)$$

The corresponding contribution to the current reads

$$j_\mu^{(subst)}[A] = -tr[\gamma_\mu G^{(subst)}(x)] = \frac{ie}{4\pi} \frac{m}{|m|} \epsilon_{\mu\nu\alpha} \partial_\nu A_\alpha(x). \quad (68)$$

The complete expression for the the fermionic current (up to order e and up to first derivatives of the gauge field) can now be obtained by adding $j_\mu^{(local)}$ in eq.(59) and $j_\mu^{(subst)}$ in eq.(68). The answer reads

$$j_\mu^{reg} = \frac{ie}{4\pi} (\pm 1 + \frac{m}{|m|}) \epsilon_{\mu\nu\alpha} \partial_\nu A_\alpha(x). \quad (69)$$

In order to compare this result with those obtained in previous works, let us first briefly insist on its domain of validity. Original works showing parity violation and the consequent emergence of a CS term for massless and massive fermions were one-loop calculations performed in a constant $F_{\mu\nu}$ background. For massless fermions we were able in I, through the ζ -function approach, to evaluate the *complete* parity violation contribution for *arbitrary* $F_{\mu\nu}$ background. This contribution reappears now in the massive case and

corresponds to the first term in eq.(69). Again this contribution corresponds to an *exact* result (not just a one-loop contribution) valid for *arbitrary* A_μ . Now, in the massive case one has also to consider the contribution given by $G^{(subst)}(x)$, i.e. the parity violation coming from the fermion mass term. This, we were not able to evaluate in a closed form due to the impossibility of calculating the Dirac Green function in an arbitrary background. For that reason, we had to appeal to some sort of approximation. We choosed a perturbative calculation still valid for arbitrary $F_{\mu\nu}$ and we stopped at the first order in e . We also neglected $O(\partial^2/m^2)$ in a derivative expansion. Of course our perturbative expansion can be systematically employed to calculate higher order contributions to the second term in eq.(67). Although we made the explicit calculations for Abelian gauge fields, the ζ -function approach can be used with no further modifications in the non-Abelian case.

Eq.(69) is the main result in our paper and we shall devote the next section to the discussion of its origin and implicancies.

4 Discussion

Through the presence of a term proportional to the Levi-Civita pseudo-tensor in the ground state fermionic current, eq.(69) shows that parity is violated when massive fermions are considered in three dimensional space time. This in turn produces a Chern-Simons term in the effective action for the gauge field.

A first important point to remark in formula (69) concerns the existence of two clearly differenciaded sources of parity violation: one is related with the fact that a mass term neccesary violates P and T in 3 dimensions. This fact leads to the second term in (69). In contrast, the first term in (69) is originated in peculiarities of regularization in odd-dimensional space-times and has nothing to do with the fermion mass. It is a pure ultraviolet effect (it arises from the necessity of making regular the singular behavior of the Green function at $x = y$) while the second term in (69) can be interpreted as an infrared contribution which cannot be naively extended to the $m = 0$ case. In fact, we proved in I that this term is absent if one starts with $m = 0$.

Once one gets the ζ -function answer to the parity violation term in the fermionic current, it is natural to try to understand the disagreement of the corresponding result with those obtained in refs.[6]-[7]. First, we note that

for $m = 0$ there is no disagreement at all, except for the interpretation of the origin of the anomaly. Indeed, quantitatively, the answer in [6]-[7] and in I was the same. Now, while the double sign origin was traced back in the former references to the necessity of introducing regulating mass, it was clear in the ζ -function approach, where no mass terms is introduced by hand, that it arises from properties of odd dimensional spaces.

When the fermions are massive, there are quantitative discrepancies apart from interpretation differences. Indeed, the ζ -function answer contains an additional contribution not taken into account for example in [7]. This situation much resembles what happens in even dimensional spaces concerning gauge anomalies: also in that case different regularization schemes lead to two different results for the anomaly in the gauge current conservation equation. One is known as the covariant anomaly, the other one as the consistent anomaly (the ζ -function approach leading automatically to the covariant result; a particular heat-kernel approach leading for example to the consistent one [17]). As explained in [26] it is on physical grounds that one should decide to which result attach or, in other words, which regularization scheme one has to adopt.

Keeping this analogy in mind, we conclude that the particular physical situation will determine which result one should use for the parity anomaly. We think that our treatment has shown, however, the unnaturalness of those proposals in which, starting with an even number of fermions and then choosing half the masses with one sign, half with the other, parity conservation is achieved: there is a parity violating effect which is intrinsic to odd dimensions and which cannot be accommodated, in our opinion, by a clever choice of mass signs. It is there to remain and it should serve as a guide even when other regularization prescriptions are adopted.

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